RESEARCH ARTICLE OPEN ACCESS

Measuring Robustness on Generalized Gaussian Distribution

Hyeon-Cheol Lee
Payload Electronics Team, Korea Aerospace Research Institute,
169-84, Gwahak-ro, Yuseong-gu, Daejeon 305-806, Rep. of Korea

ABSTRACT
Different from previous work that measured robustness its own distribution, measuring robustness with a robust estimator on a generalized Gaussian distribution is introduced here. In detail, an unbiased Maximum Likelihood (ML) variance estimator and a robust variance estimator of the Gaussian distribution with a given censoring value are applied to the generalized Gaussian distribution that represents Gaussian, Laplace, and Cauchy distributions; then, Mean Square Error (MSE) is calculated to measure robustness. Afterward, how robustness changes is shown because the actual distribution varies over the generalized Gaussian distribution. The results indicate that measuring the MSE of the system can be used to point out how robust the system is when the system distribution changes.
Keywords – generalized Gaussian distribution, MSE, robustness, ML estimator, robust estimator

I. Introduction

Many of the techniques used in telecommunications, image, speech, and radar signal processing rely on various degrees of measuring system performance and robustness. To measure overall system performance, the degree of robustness, which shows how stable a system is, is very important.

A considerable contribution to robustness using saddle point criteria has been made by Huber [1-3], who censored the height. His work, however, was nonquantitative, did not easily admit nonstationary data, and gave no direct way to make comparisons between multiple systems. Lee and Halverson [4] presented a differential geometric approach that admitted nonstationary data and offered quantitative robustness measurement using a robust estimator (the Huber estimator) for its own distribution. Petrus [5] used Huber adaptive filters to reject impulse noises, Gaussian noises, and undesired sinusoidal signals.

In this paper, a Maximum Likelihood (ML) variance estimator (typically unbiased) [6, 7] and a robust variance estimator [1-3] are recalled for robustness and then applied to the generalized Gaussian distribution family and the robust estimator measures Mean Square Error (MSE) according to the censoring of the height k value. After that, the MSE is plotted over the generalized Gaussian (light and heavy-tailed) distributions that represents Gaussian, Laplace, and Cauchy distributions, and the way in which performance (= inverted MSE) and robustness vary with each k according to changes of the distribution is found.

This paper is organized as follows. Section 2 will illustrate the notion of performance and robustness. Section 3 will show the measuring of MSE. Section 4 will introduce the generalized Gaussian distribution and MSE graphs. Section 5 will conclude.

II. Notion of performance and robustness

The notion of performance can best be understood by noting that an estimator generates a multidimensional MSE surface. Fig. 1 illustrates the x-axis, which indicates an estimation value, $\hat{\theta}$ of the distribution; the y-axis indicates the MSE, $E(\theta-\hat{\theta})^2$. If $\hat{\theta}$ approaches a true value of θ (variance in this paper), the MSE reaches its minimum. The ML estimator generates an MSE surface resembling an inverted mountain; on the contrary, a robust processor might generate an MSE surface resembling an inverted plateau. It should be noted that if $\hat{\theta}$ is sufficiently far away from θ, then the robust estimator can have an MSE smaller than that of the ML estimator. In other words, a shift away from θ will only bring about a small MSE for the robust estimator of this type, but the ML estimator has a large MSE.

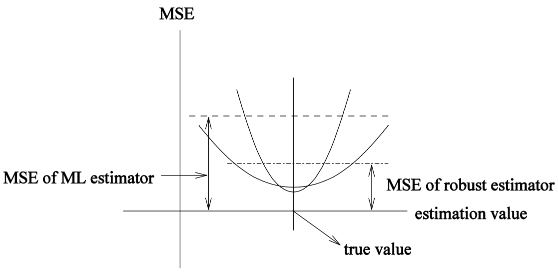


Fig. 1 MSE surface of ML and robust estimator

Robustness [8, 9] can be understood by considering Fig. 1 again. For calculation of the robustness, the slope of the multidimensional MSE surface in Fig. 1 is converted to quantitative values using a differential geometric approach [10]. To determine the measure of robustness, it is necessary to admit a variation rate in the distribution of the parameter. Good robustness means a small change of output as input varies, resulting in a gentle MSE curve. This tells us that measuring the robustness indicates the stability of a system.

III. Measuring of MSE

In this section, how to calculate the MSE in cases of Gaussian distribution with an ML estimator and a robust estimator [4] is shown.

3.1 Mathematical derivation of MSE

Consider the parameter estimation [3, 4] of θ and an estimator $\hat{\theta} = g(\cdot, \cdots, \cdot)$ with $E\{Q(\theta - \hat{\theta})\}$. There are n independent samples. Q(·) could be a mean square error or a mean absolute error function; the MSE is used for this paper. The MSE is expressed in the following form for nominally stationary data ($0 \le MSE \le \infty$):

$$E(\theta - \hat{\theta})^2 = \int_{R^n} (\theta - \frac{1}{n}\sum_{j=1}^{n} h(x_j))^2 f(x_1)$$

$$\cdots f(x_n) dx_1 \cdots dx_n , \qquad (1)$$

which is recalled from Equation (27) of [4], expressed using the following form again

$$E(\theta - \hat{\theta})^2 = \theta^2 - 2\theta \int_{-\infty}^{+\infty} h(x) f(x) dx + (1 - \frac{1}{n})$$

$$(\int_{-\infty}^{+\infty} h(x) f(x) dx)^2 + \frac{1}{n}\int_{-\infty}^{+\infty} h^2(x) f(x) dx , \qquad (2)$$

where f(x) is the univariated density of the nominal i.i.d. data, the Gaussian distribution in this paper. Taking the inverse of the MSE is performance. Function h(x) will be explained in (4), later.

3.2 ML and robust estimator of Gaussian distribution

The estimators under consideration are derived from, whenever possible, unbiased ML estimators, which are not robust. In order to induce a robust estimator, Huber-type data censoring is employed. In accordance with the original estimator being unbiased for cases of Gaussian data, the variance is estimated when the data are nonstationary but independent.

Consider the zero-mean Gaussian distribution, the unbiased ML variance estimator is

$$\hat{\theta} = \frac{1}{n}\sum_{j=1}^{n} x_j^2, \quad h(x) = x^2 . \qquad (3)$$

The robust estimator with h(x) is given by

$$\hat{\theta} = \frac{1}{n}\sum_{j=1}^{n} h(x_j) , \quad h(x) = \begin{cases} k^2, |x| > k \\ x^2, |x| \le k \end{cases} . \qquad (4)$$

The specific integral parts of the MSE for the robust estimator recalled from (28) and (29) of [4], are

$$\int_{-\infty}^{+\infty} h(x) f(x) dx$$

$$= 2\int_0^k x^2 f(x) dx + 2\int_k^{+\infty} k^2 f(x) dx \,. \tag{5}$$

$$\int_{-\infty}^{+\infty} h^2(x) f(x) dx$$

$$= 2\int_0^k x^4 f(x) dx + 2\int_k^{+\infty} k^4 f(x) dx \,. \tag{6}$$

Where f(x), the Gaussian distribution, is

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right). \tag{7}$$

Computation of (2) with these equations is possible using MAPLE.

IV. Generalized Gaussian distribution

4.1 Understanding of generalized Gaussian distribution

The first step here is to consider situations with known variance and to compute actual MSE values for the estimators, because the underlying distribution varies away from the nominal Gaussian distribution due to the aforementioned known variance. The key to obtaining tractable results here is to model the variations in a way that admits computation of the MSE and yet allows a smooth morphing of distributions away from the Gaussian in directions that are both more heavy-tailed and more light-tailed. To do this, the family of zero-mean generalized Gaussian distributions is chosen, i.e., a family with density of the form

$$f(x) = d \cdot \exp\left(-\frac{|x|^a}{c}\right). \tag{8}$$

It should be noted that f(x) is a Gaussian distribution at a = 2, a Laplace distribution at a = 1, and resembles a Cauchy distribution when a approaches 0. For a > 2, the distribution becomes increasingly light-tailed, and for a < 2 it is increasingly heavy-tailed. The heavy-tailed densities are expected to place stress on the estimators; as a result, more values of a < 2 than a > 2 will be considered. It should be noted that with this generalized Gaussian distribution a model that can pair distributions with real numbers a > 0 is shown. In order to measure the MSE of the generalized Gaussian distribution, (8) is used instead of (7). Computation of (2) with (5), (6), and (8) is then possible using MAPLE.

4.2 Plotting MSE versus constant a of generalized Gaussian distribution

The MSE is plotted as a function of the real number a; specific examples of the generic graph are illustrated in Fig. 1. For an equitable comparison constants c and d are chosen so that as a varies, θ ($= \int_{-\infty}^{+\infty} x^2 f(x) dx$) of the random variable is kept constant.

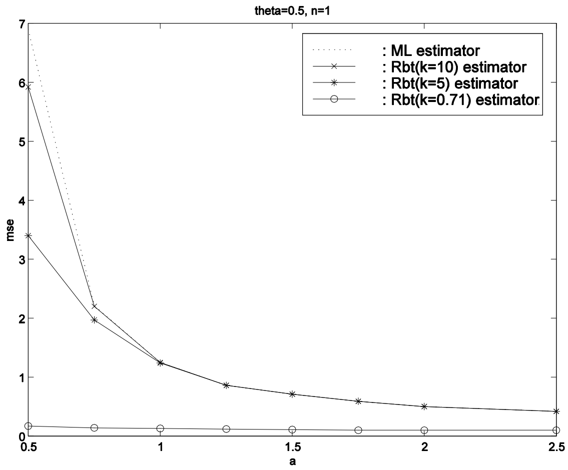
For fixed actual variance θ, the MSE is plotted as a function of constant a and it can be seen how flat the graphs are; as indicated in Section 2, steep slopes indicate lack of robustness. It should also be remarked that any comparison of the graph of one estimator to another should be made with the understanding that the robust estimators have bias, and so even though the ML estimator for Gaussian data is efficient, its MSE may not be minimal when compared to the biased estimators. The procedure is, for fixed a and θ, to search for corresponding c and d that yield a variance of θ for the operative distribution; then, it is necessary to use these c and d values to compute the MSE for the estimators. Because of the sensitivity of θ to c and d, the search should stop when θ has been approximately achieved. Table 1 illustrates the choices of c and d for various a and for θ = 0.5, 1.0, 2.0, and 5.0.

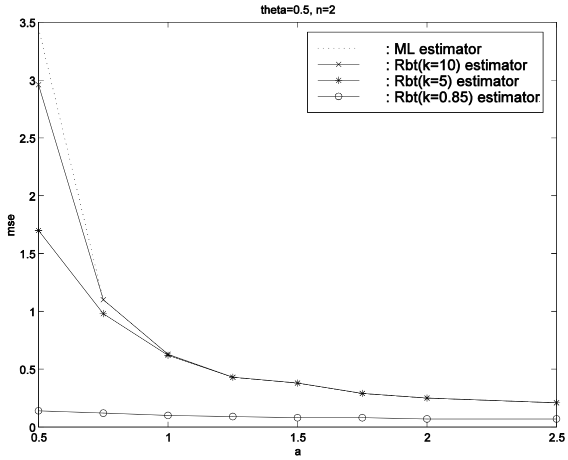
Table. 1 Coefficients of generalized Gaussian distribution

	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50
θ	0.5							
c	0.26	0.38	0.50	0.62	0.76	0.88	1.00	1.30
d	3.50	1.50	1.00	0.78	0.65	0.60	0.56	0.50
θ	1.0							
c	0.30	0.48	0.71	0.96	1.30	1.60	2.00	3.10
d	2.78	1.12	0.70	0.55	0.45	0.43	0.40	0.35
θ	2.0							
c	0.38	0.15	1.00	1.50	2.20	3.00	4.00	7.50
d	1.77	0.72	0.50	0.38	0.32	0.30	0.28	0.25
θ	5.0							
c	0.45	0.90	1.60	2.60	4.10	6.40	10.0	24.0
d	1.25	0.48	0.31	0.25	0.22	0.20	0.18	0.15

4.3 Measuring Robustness

With constant values chosen in this way, the MSE is calculated using (2). Fig. 2 - Fig. 5 illustrate plots for the MSE versus a with the same θ. In each graph, the value a = 2 should be regarded as nominal, because the estimator was designed using a Gaussian model. It should be noted that in all cases there is more variability in the MSE as a becomes smaller, i.e., as the data become more heavy-tailed.





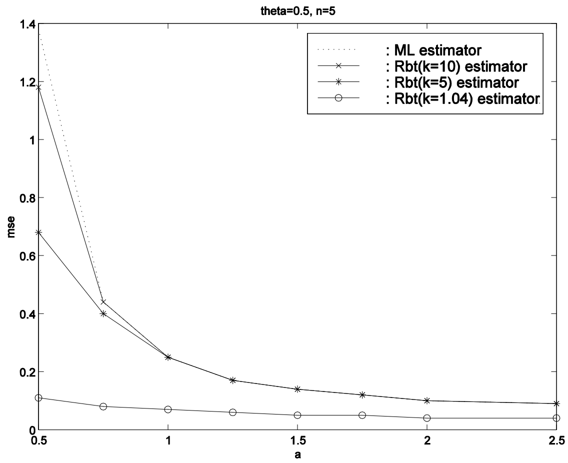
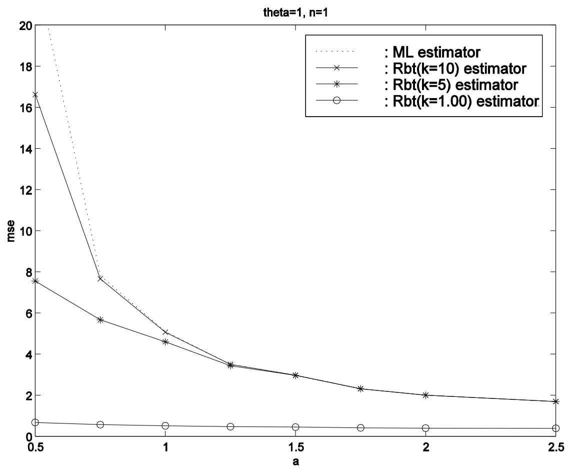


Fig. 2 MSE of Generalized Gaussian distribution, θ= 0.5.



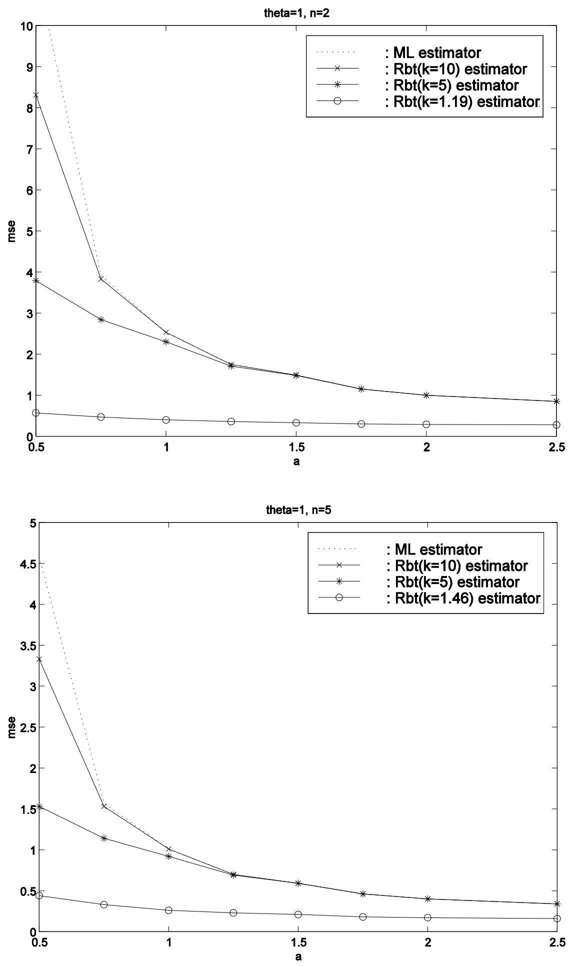


Fig. 3 MSE of Generalized Gaussian distribution, θ= 1.0.

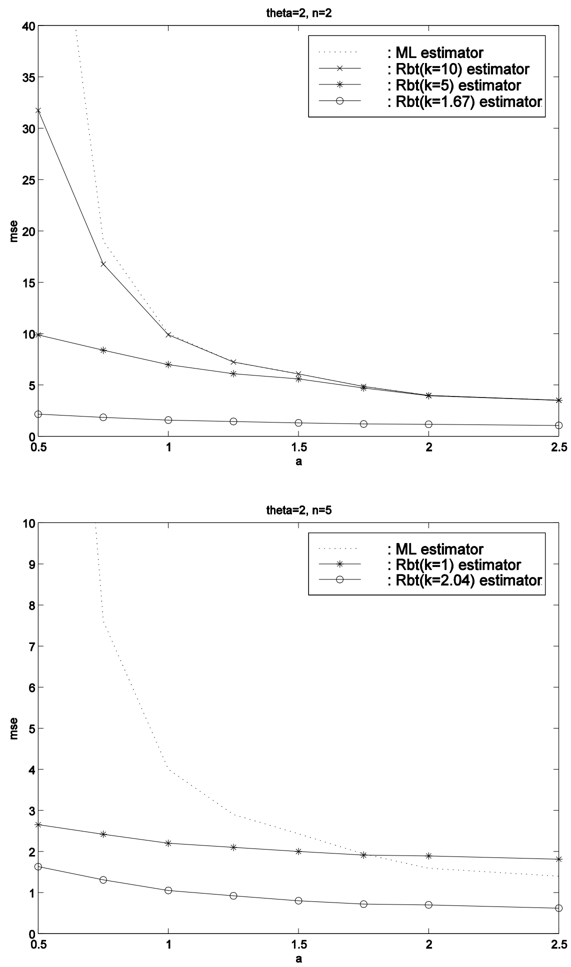
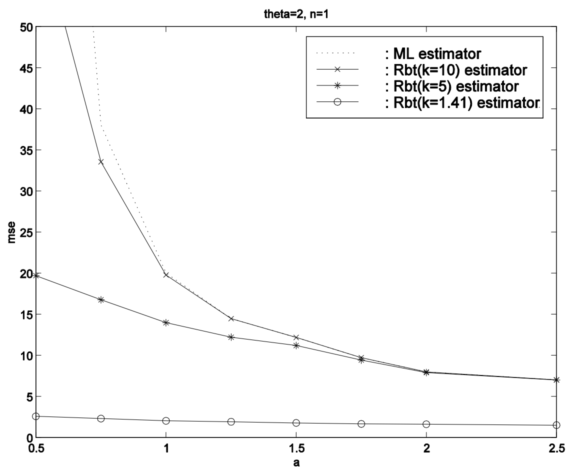
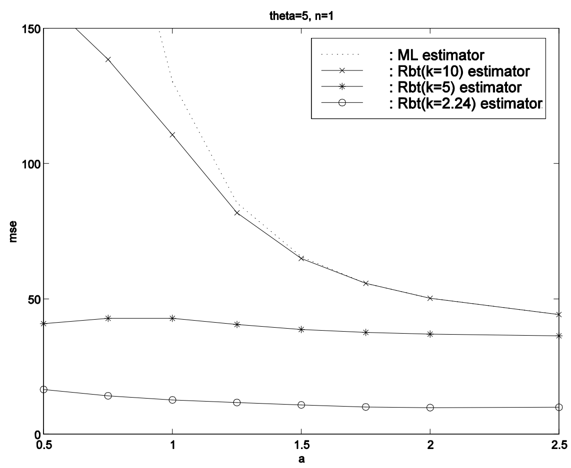


Fig. 4 MSE of Generalized Gaussian distribution, θ= 2.0.





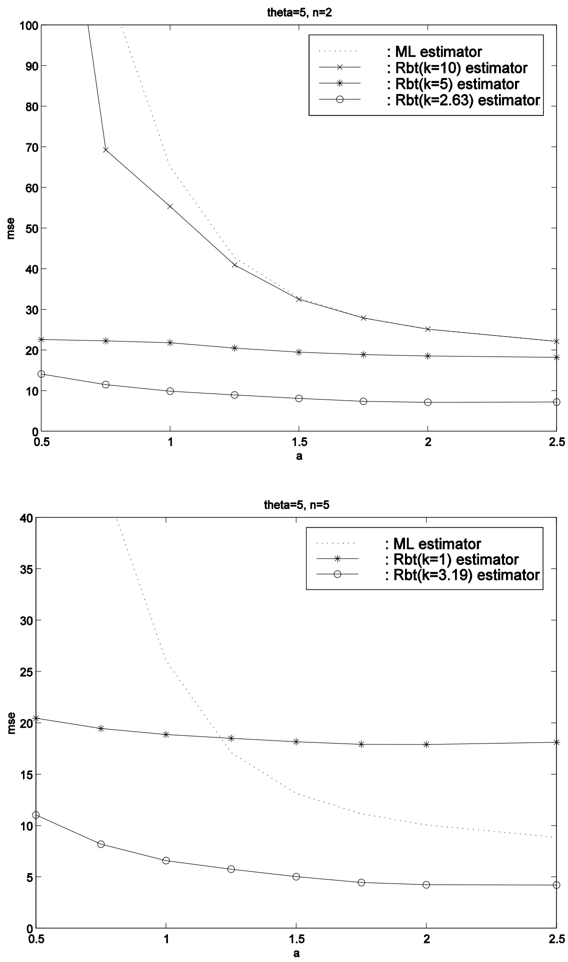


Fig. 5 MSE of Generalized Gaussian distribution, θ= 5.0.

In addition, for heavy-tailed data the MSE becomes larger than for the nominal case. On the other hand, for light-tailed perturbations the MSE is not highly variable and decreases from the nominal MSE value. This confirms the expectation that heavy-tailed data put great stress on the estimator, in terms of both performance and robustness (quantitative values of robustness are not shown here), whereas light-tailed perturbations typically present little problem for the estimator.

The graphs indicate that if a considerable lack of knowledge exists in the heavy-tailed direction, i.e., if strongly heavy-tailed perturbations are regarded as possible, then heavy censoring (small k) should be employed when substantial lack of robustness is present, such as when n is small. But, in other cases, such as for large n, much less MSE variability is seen, and in these situations less censoring (large k) is appropriate in order to avoid compromising performance.

Measuring the robustness of a system with the chosen k value of the robust estimator is possible. For θ = 1, n = 5, and k = 1.46, if the MSE of the system output is larger than 0.5, the system is not robust.

It should be remarked that the graphs should not be used to compare k values directly because of the varying bias, which can compromise the field performance of the estimator. The important thing is the shape of each curve the flatter the curve the more robustness there is, and so if the curves are relatively parallel for many choices of k, then it makes sense to choose a larger value of k, which will bring us closer to the unbiased ML estimator. For n = 5 shown in Fig. 4 and Fig. 5, k = 1 is over censored.

V. Conclusion

The previous work measured robustness on its own distribution; however, measuring robustness with a robust variance estimator by censoring height on a generalized Gaussian distribution is introduced in this paper. The MSE is measured with an unbiased ML variance estimator and a robust variance estimator of the Gaussian distribution on the generalized Gaussian distribution. Then, the MSE is plotted according to the constant of the generalized Gaussian distribution which represents Gaussian, Laplace, and Cauchy distributions. This measuring of MSE with the given censoring value indicates the system robustness which tells us the stability of a system when the system distribution changes.

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